Serre's GAGA Principle

An Accessible Introduction

Sam Brunacini

Math 2501
University of Pittsburgh
December 18, 2023

1 Introduction

Géométrie Algébrique et Géométrie Analytique (GAGA) is a powerful principle allowing for the application of algebraic methods to analytic spaces and vice versa. The main result is Serre's GAGA Theorem, which establishes the equivalence of two categories that we will define later. The goal of this paper is to describe this important relation in the most accessible manner possible. To that end, we assume only a background in commutative algebra, elementary topology, and an understanding of the affine Nullstellensatz.

As an application, we look briefly at a corollary called Chow's Theorem. This is among the most immediately useful implications of GAGA: it tells us about a large class of complex projective spaces that can be viewed as the zero locus of a set of functions.

2 Background

In order to clearly state the GAGA Theorem, we must define some terms in algebraic geometry. These definitions are phrased to be sufficient for the goal of understanding the theorem. As a result, they may not be fully general. We consider only commutative rings with unity.

Definition A *sheaf* F on a topological space X associates to each inclusion of open sets $U' \subset U \subset X$ a set $F(U)$ with elements (called *sections*) and a set of "restriction morphisms" $F(U) \to F(U')$. We write the image of $f \in F(U)$ under such a map as $f\big|_{U'}$. We require these morphisms to satisfy certain properties so that sections mimic the behavior of functions. In particular, the restriction of $F(U)$ to U is the identity and restrictions compose as expected. That is, restricting $F(U)$ to $F(U')$ and then to $F(U'')$ where $U'' \subset U'$ should be the same as the restriction of $F(U)$ directly to $F(U'')$. Furthermore, for any open sets U_i and U_j in an open cover of a subset U and sections f_i and f_j in $F(U_i)$ and $F(U_j)$ respectively: If $f_i\big|_{U_i \cap U_j} = f_j\big|_{U_i \cap U_j}$, then there is a unique section $f \in F(U)$ that agrees with f_i when restricted to U_i and f_j when restricted to U_j. This is called the unique gluing property.

If desired, we can define additional structure on the sets $F(U)$. If we give them a ring structure, for example, we refer to F as a sheaf of rings. Similarly, we can define a sheaf of modules. Suppose we have on a space X both a sheaf of rings F and a sheaf of modules G. If $G(U)$ is a $F(U)$-module for each $U \in X$, we say G is an F-module.

Example The association O_X of each open subset $U \subset \mathbb{C}^n$ to the ring of holomorphic functions on U is a sheaf of rings.

Definition A *ringed space* is the pair (X, O_X) where X is a topological space and O_X is a sheaf of rings on X called the structure sheaf. A ringed space is *local* if the structure sheaf is a local ring in some neighborhood of each point $x \in X$. In this case, we call it a locally ringed space.

Definition A scheme is a locally ringed space (X, O_X) such that for U in an open cover U_i of X, U (viewed as a ringed space) is isomorphic to $\operatorname{Spec} R$ for some ring R. We say the scheme is of finite type over S if each R_i is finitely presented over S.

Example A locally ringed space isomorphic to $\operatorname{Spec} k[x_1, \cdots, x_n]$ is a scheme. In fact, schemes that are themselves isomorphic to the spectrum of a ring are given a special name: affine schemes. $\mathbb{A}_n^k$ is sometimes defined this way.

Definition A *coherent sheaf* on a locally ringed space (X, O_X) is a finitely generated O_X-module.

Definition Complex analytic spaces are locally ringed spaces that are locally isomorphic to an analytic variety. The notion of analytic variety is analogous to that of an algebraic or projective variety. That is, it is the zero locus of some ideal generated by a set of analytic functions. We denote the zero locus of the ideal of functions I as $Z(I)$.

Examples $Z((x))$ is an algebraic, projective, and analytic variety. Even $Z((\exp(x)))$ is an algebraic variety (the empty set) even though the exponential function has no representation as a finite polynomial. Consider $Z((\sin(x)))$. On the real line, $\sin(x)$ has zeros at $n\pi$ for all integers n. This immediately proves that this cannot be an algebraic variety since the level set $\operatorname{Re}(\sin z) = 0$ defines a one-variable real function with countably infinitely many zeros, which obviously cannot be a polynomial.

3 Analytification

In order to transfer results back and forth between an algebraic and analytic context, we need a way to talk about complex schemes as analytic spaces. We also want to be able to transfer statements about coherent sheaf structure. The process of constructing an analytic space X^{an} from the scheme X is called analytification. Similarly, we write F^{an} to denote the analytification of a coherent sheaf F on X.

3.1 Spaces

Let X be a scheme of finite type over $\mathbb{C}$. Since X is a locally ringed space, we can define an open cover Y_i of X with $Y_i \simeq \operatorname{Spec} R_i$ for rings R_i. X is of finite type, so $R_i \simeq \mathbb{C}[x_{1i}, \ldots, x_{ni}]/(f_{1i}, \ldots, f_{mi})$ for some set of polynomial generators $f_{1i}, \ldots, f_{mi} \in \mathbb{C}[x_{1i}, \ldots, x_{ni}]$.

We define the *analytification* of X as the gluing of the analytic variety associated with each ideal of functions $(f_{1i}, \ldots, f_{mi})$, when these polynomials are regarded as holomorphic functions.

In this way, we obtain a functor

$$h : \text{Schemes of finite type over } \mathbb{C} \to \text{Complex analytic spaces.}$$

We call this the "analytification functor" [3]. Even without GAGA, we can make some statements about its properties. For example, if we have a short exact sequence of analytic spaces that are expressible as analytifications, the pullback of this sequence along h is still exact. In other words, h is flat as a morphism of locally ringed spaces [4].

3.2 Coherent Sheaves

We can use the pullback of a coherent sheaf F along the natural map of ringed spaces $(X^{an}, O_X^{an}) \to (X, O_X)$ to define F^{an} [3, 4]. The nature of this correspondence is the main information the GAGA Theorem tells us.

We are left with questions about the behavior of h. First of all, What structure does it preserve? This is a big question, but to state a few properties: h preserves connectedness (although the topology changes), and X is smooth if and only if X^{an} is a manifold [3]. Can any analytic space be written as an analytification? What about the same question for coherent sheaves on analytic spaces? As it turns out, the answer to these questions is no in general.

Counterexample The complex unit disk cannot be expressed as an analytification [5].

However, if we focus our consideration only on projective schemes, this changes. Finally, we can state GAGA.

4 Main Statement

Let $\underline{\mathrm{Coh}\,X}$ denote the category of coherent sheaves on the space X.

Theorem: GAGA (Serre) The analytification functor h induces an equivalence of the categories $\underline{\mathrm{Coh}\,X}$ and $\underline{\mathrm{Coh}\,X}^{\mathrm{an}}$, where X is a complex projective scheme.
Note: the full statement of the theorem gives a result about the cohomology of coherent sheaves, but it is beyond the scope of this paper.

As a brief note on equivalence of categories: the theorem implies there is a functor $h^{-1} : \underline{\mathrm{Coh}\,X}^{\mathrm{an}} \to \underline{\mathrm{Coh}\,X}$ such that each coherent sheaf on X is isomorphic to its image under the composition of functors $h^{-1} \circ h$. This does not mean $h^{-1} \circ h$ is the identity functor. The important takeaway from this is not in the details. Instead, this equivalence means, loosely speaking, that we can apply theorems about either category to **both** categories, an extremely powerful result.

This answers affirmatively the last question asked in the previous section in the case of complex projective schemes. The answer to the remaining question does not follow as immediately. However, it still turns out to be true. This leads us to Chow's Theorem.

5 Implications

Chow's Theorem. Statement: Hartshorne
If $\mathfrak{X}$ is a compact analytic subspace of the complex manifold $\mathbb{P}^n_{\mathbb{C}}$, then there is a subscheme $X \subset \mathbb{P}^n_{\mathbb{C}}$ with $X^{\text{an}} = \mathfrak{X}$

In other words, we can look at any complex projective subspace satisfying the theorem's conditions (which, notably, are very weak!) as the zero locus of some set of polynomials, allowing for the direct application of algebraic methods.

6 Acknowledgements

Special thanks to Dr. Carl Wang-Erickson for the topic idea for this paper and assistance while trying to narrow down its scope, and to Elijah Van Vlack, Yiqing Wang, Christopher Manseau, and Matthew Snodgrass for providing useful feedback for refining the ideas and scope of the paper.

References

[1] "Affine Scheme." nLab, ncatlab.org/nlab/show/affine+scheme. Accessed 17 Dec. 2023.

[2] Halleck-Dube, Connor. GAGA Talk Notes, University of California, Berkeley, math.berkeley.edu/~chd/expo/GAGA.pdf. Accessed 17 Dec. 2023.

[3] Hartshorne, Robin. Algebraic Geometry, Springer, New York, New York, 1997, pp. 438–445.

[4] Kedlaya, Kiran. GAGA, Massachusetts Institute of Technology, ocw.mit.edu/courses/18-726-algebraic-geometry-spring-2009/1329d3923b9f2d55ec7a9ec8e9cb63e9_MIT18_726s09_lec22_gaga.pdf. Accessed 17 Dec. 2023.

[5] McKernan, James. Model Answers to HWK #2, University of California, San Diego, mathweb.ucsd.edu/~jmckerna/Teaching/13-14/Spring/203C/model2.pdf. Accessed 18 Dec. 2023.

[6] "Sheaf of Modules." Wikipedia, Wikimedia Foundation, 13 Dec. 2023, en.wikipedia.org/wiki/Sheaf_of_modules#Sheaf_associated_to_a_module.

[7] user144221. "The Global Sections Functor and the Hom Functor." Mathematics Stack Exchange, 12 Jan. 2017, math.stackexchange.com/questions/2094400/the-global-sections-functor-and-the-hom-functor.