Serre's GAGA principle

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Introduction

GAGA is a principle allowing for the application of algebraic methods to analytic spaces. The main result is Serre's GAGA Theorem, which establishes the equivalence of two categories that we'll define later.

As an application, we look at a corollary called Chow's Theorem.

Sheaves of rings: A very case-specific definition

This is by no means the definition of a sheaf in full generality - just a sufficient definition for the purposes of this presentation.

Definition A sheaf F on a topological space X associates to each inclusion of open sets $U' \subset U \subset X$ a set $F(U)$ with elements (called sections) and a set of "restriction morphisms" $F(U) \to F(U')$ that satisfy a few properties to behave like functions.
In particular, the restriction of $F(U)$ to U is the identity and restrictions compose as expected. Furthermore, for sets U_i forming an open cover of a subset U and sections f_i in $F(U_i)$: If $f_i\big|_{U_i \cap U_j} = f_j\big|_{U_i \cap U_j}$, then there is a *unique* section $f \in F(U)$ that agrees with f_i when restricted to U_i.

Example: Sheaf of Rings The association O_X of each open subset $U \subset \mathbb{C}^n$ to the ring of holomorphic functions on U is a sheaf.

Ringed Spaces and Schemes

Definition A ringed space is the pair (X, O_X) where X is a topological space and O_X is a sheaf of rings on X called the structure sheaf. A ringed space is *local* if each of the structure sheaf's stalks (sheaf equivalent of germ) is a local ring.

Definition A scheme is a locally ringed space (X, O_X) such that for U in an open cover U_i of X, U (viewed as a ringed space) is isomorphic to $\operatorname{Spec} R$ for some ring R. We say the scheme is of finite type over S if each R_i is finitely presented over S.

Example $\mathbb{A}_k^n$ is a scheme.

Coherent Sheaf: Notion of finitely generated O_X-module

Complex Analytic Spaces

Definition Complex analytic spaces are locally ringed spaces that are locally isomorphic to an analytic variety. Note: Loosely speaking, the notion of analytic variety is directly analogous to that of the algebraic variety that we discussed in class. That is, it is the zero locus of some ideal generated by a set of analytic functions. (We can ignore its sheaf structure)

Example $Z((x))$

Analytification

Let X be a scheme of finite type over $\mathbb{C}$. Since X is a locally ringed space, we can define an open cover Y_i of X with $Y_i \simeq \operatorname{Spec} R_i$ for rings R_i. X is of finite type, so $R_i \simeq \mathbb{C}[x_{1i}, \ldots, x_{ni}]/(f_{1i}, \ldots, f_{mi})$.
Note that X is the gluing of Y_i. We define the *analytification* of X as the gluing of the zero loci of each ideal of functions $(f_{1i}, \cdots, f_{mi})$ (when these polynomials are regarded as holomorphic functions).

In this way, we obtain a functor
h : Schemes of finite type over $\mathbb{C} \to$ Complex analytic spaces. We call this the "analytification functor".

Properties of Analytification

What is preserved by h?

Can an arbitrary complex analytic space be written as X_{an} for some scheme X?

Theorem: Equivalence of categories

So, we consider the projective case. More specifically, we can restrict our view to complex projective schemes. Then, we can establish a tighter relation:

Theorem: GAGA (Serre) The analytification functor induces an equivalence of the categories $\operatorname{Coh} X$ and $\operatorname{Coh} X^{\mathrm{an}}$, where X is a complex projective scheme.
Note: the full statement of the theorem gives a result about cohomology, but it's beyond the scope of this project.

Consequences: Chow's Theorem

Chow's Theorem. Statement: Hartshorne
If $\mathfrak{X}$ is a compact analytic subspace of the complex manifold $\mathbb{P}^n_{\mathbb{C}}$, then there is a subscheme $X \subset \mathbb{P}^n$ with $X^{\mathrm{an}} = \mathfrak{X}$

References

Hartshorne (Appendix B)
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