Poincaré Disk Image Tiling

Painter Fellowship Undergraduate Research

Sam Brunacini
Mentor: Dr. Jason DeBlois

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Background

Poincaré Disk

The Poincaré disk is a conformal model of hyperbolic geometry in the open unit disk D in the complex plane, centered at the origin by convention.

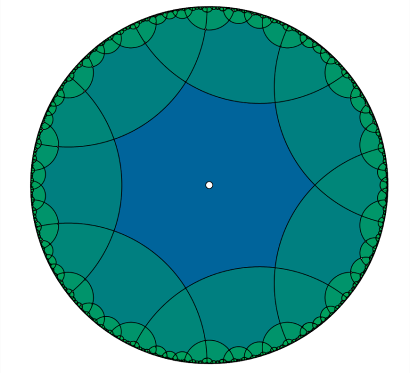
Geodesics in the Poincaré disk are circular segments orthogonal to the boundary of the disk and diameters of the disk.

The (orientation-preserving) isometries of D are elements of $\mathrm{PSU}(1)$. Similar triangles are congruent in hyperbolic geometry.

Our project

Goal

Construct interactive (p, q) tessellations of the hyperbolic plane and fill the p-gons with minimally distorted copies of a given (square) input picture.



Basic Algorithm (for triangulation-based shader)

1. Start with a square picture and parameters (p, q) and consider triangle vertex $\mathbf{x}$.
2. Send $\mathbf{x}$ to a regular p-gon via some minimally-distorting map f.
3. Center the resulting p-gon at the origin and scale so its radius corresponds to the unique rotationally symmetric regular hyperbolic p-gon in the Klein disk.
4. Map to the Poincaré disk using the hemisphere model as intermediate.
5. Tile the plane using reflections (inversion through geodesic edges) or rotations of this polygon.

Basic Algorithm (for pixel-based shader)

1. Start with a square picture and parameters (p, q) and consider the point $\mathbf{x}$ inside the rotationally symmetric p-gon in the Poincaré disk.
2. Map to the Klein disk using the hemisphere model as intermediate.
3. Send the result to the square picture using the inverse map f^{-1}.
4. Color the point $\mathbf{x}$ accordingly.
5. Tile the plane using reflections or rotations.

The map f

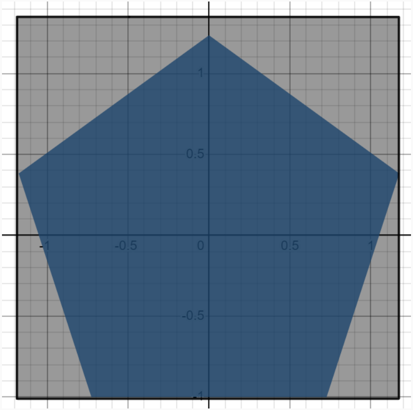
It remains to select a map f that results in the least amount of distortion.

Ideal properties:

- Minimal area distortion
- Minimal angle distortion
- Continuity
- Symmetry

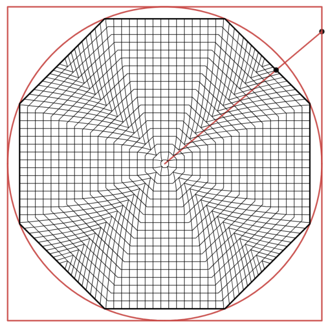
The map f: cutting out the shape

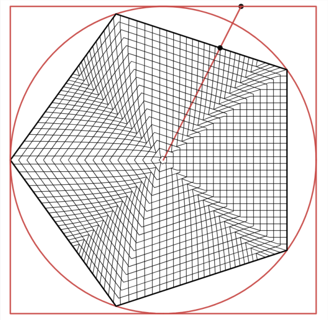
We could simply cut out the largest regular p-gon that fits into the square image and discard the rest.



The map f: radial projection

We could project the boundary of the square to the p-gon boundary and interpolate radially.

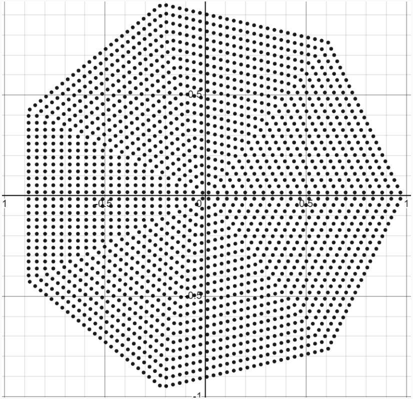




Easing (such as smoothstep or Weierstrass transform (convolving with Gaussian)) helps but does not fix these issues.

The map f: concentric mapping

We could map concentric squares to concentric p-gons one ring at a time.



This is the second best option, and the closest to being equiareal. Note this is not the same as radial projection.

The map f: Schwarz-Christoffel mapping

By the Riemann Mapping Theorem, there is a biholomorphic (and therefore conformal) mapping between the open unit disk and any simply connected open subset of $\mathbb{C}$. This application of the Schwarz-Christoffel map and its inverse are constructions of such a map for the disk and regular polygons.

Schwarz-Christoffel (disk to p-gon):

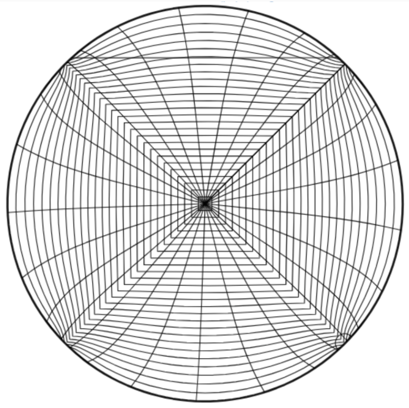
$$\int_0^z \frac{dw}{(1 - w^p)^{\frac{2}{p}}}$$

Inverse Schwarz-Christoffel (p-gon to disk):

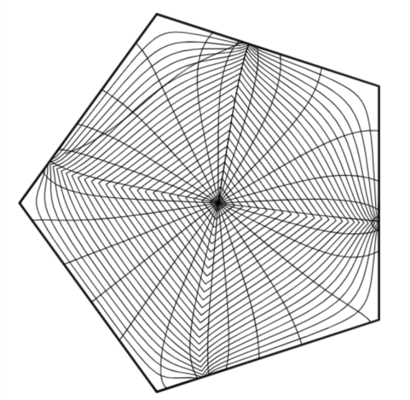
$$z \sum_{k=0}^{\infty} \frac{z^{pk}}{(pk+1)!} f^{(pk)}(0)^{-pk-1}$$

Our implementation of this series uses only the first few (5 to 6) terms, after which they become vanishingly small.

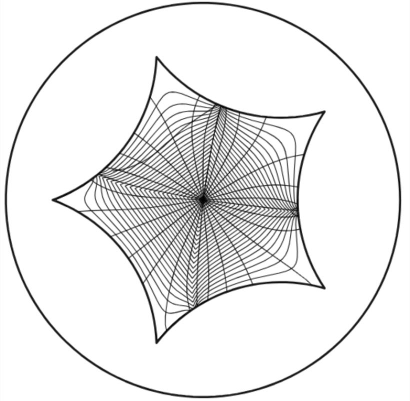
Inverse SC



SC



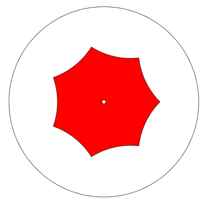
Polygon in Poincaré disk

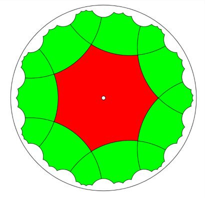


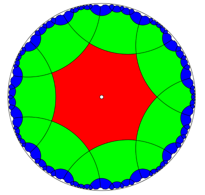
Tiling the plane

This is equivalent to the word problem for Coxeter groups. Although this is a solved problem, it is known to be computationally expensive. It is therefore more practical to use some programming workarounds.

Our program generates the tiling one "layer" at a time.







Tiling the plane

The program tracks ”outer” vertices and edges and the number of generated polygons at each outer vertex.

Once each outer vertex is saturated with q polygons, the layer is complete.

This process iterates until a specified number of layers is reached or a specified area of the disk is covered.

Github

Source code:
https://github.com/sam-lb/PoincareTessellation